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The optimization problem over a distributive lattice

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Abstract In this paper we give a necessary and sufficient condition for existence of minimal solution(s) of the linear system $A * X \geq b$ where A, b are fixed matrices and X is an unknown matrix over a lattice. Next, an algorithm which finds these minimal solutions over a distributive lattice is given. Finally, we find an optimal solution for the optimization problem $\min\{Z = C * X | A * X \geq b\}$ where C is the given matrix of coefficients of objective function Z.

Keywords Distributive lattice · Linear programming · Fuzzy linear systems · Fuzzy relational equation · Optimization

Mathematics Subject Classification (2000) 06Dxx · 34A30 · 20M14 · 16Y60

1 Introduction

Linear and combinatorial optimization have been studied by many authors. Optimization over residuated lattice-ordered commutative monoid is studied in [14]. Furthermore, in many applications, one needs to find the solution of fuzzy linear systems of equations and inequalities over a bounded chain in [8,10] in finite dimensional case and for systems with infinite number of variables [13]. The resolution problem for fuzzy relation equations has been put forward by Sanchez [11]. Following his fundamental result for the greatest solution, many other authors proposed thorough investigation for a variety of special resolution problems, using different mathematical methods. We shall list the main results in this field. Let us suppose that the fuzzy relation equation is consistent. Then:

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- it has a greatest solution [11], which is easily computable;
- the minimal solutions of a fuzzy relation equation are computable [9];
- all solutions of fuzzy relation equation are completely determined by a minimal and the greatest solutions [4,7];
- the requirements for a unique solution of a fuzzy relation equation in a complete Brouwerian lattice are investigated [12];

Obviously, there exist many interesting results obtained by many authors and with highly varied mathematical methods. First of all we would like to pay attention to the difference between the classical problem, i.e., how to solve conventional linear systems with traditional addition and multiplication and how to solve linear systems over a lattice which is the subject of this study. There is also some differences between solving a linear system over a bounded chain [8,10] and solving it over a bounded distributive lattice which will be discussed in this study. In [5] bounded chain is replaced by a pseudo-Boolean lattice for solving the linear system of inequalities $A * X \leq b$. Then, by using the approach developed in [14] authors solved the linear system of inequalities $A * X \leq b$, over a pseudo-Boolean lattice L. The method which was given there is very easy to apply for solving the fuzzy linear systems studied in [8]. Note that the existence of minimal solutions of $A * X \geq b$ over a bounded chain was proved in [6]. For L-fuzzy linear systems $A * X \geq b$ and $A * X = b$ over a bounded distributive lattice, a necessary and sufficient condition for consistency of systems was given in [6]. In this paper, we consider a distributive lattice L. First, we give a necessary and sufficient condition for existence of minimal solutions. Then, we obtain an algorithm for finding these minimal solutions when there are finitely many of them. In the last section we give an algorithm which allows to find an optimal solution for the optimization problem $\min\{Z = C * X | A * X \geq b\}$. It is interesting that in this algorithm we compute the minimum value of Z without computing all minimal solutions of the linear system of inequalities. Moreover, it can be used when the system has infinitely many minimal solutions.

2 Preliminaries

In this section we give some preliminaries which we need in sequel sections. In the next exposition the terminology for lattice theory and algebra is according to [1,3].

Definition 2.1 Let $(L, \leq)$ be a lattice and $S \subseteq L$.

(i) If $(S, \leq)$ is a lattice, then S is called a sublattice of L and denoted by $S \leq_l L$.
(ii) If $a \vee b(a \wedge b$, respectively) exists for all $a, b \in S$, then S is called join-semi-sublattice (meet-semi-sublattice, respectively) of L (see [3]).
(iii) A sublattice I is called an ideal, if $a \in I$, $x \in L$ and $x \leq a$ implies $x \in I$.
(iv) A sublattice S is called convex, if $x, y \in S$ and $x \leq z \leq y$, then $z \in S$.

Note that every convex sublattice is an ideal (see [3], p. 17).

Definition 2.2 (i) A lattice $(L, \leq)$ is called distributive if

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \,,$$

or equivalently

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \,.$$

(ii) A lattice $(L, \leq)$ is called complete if $\bigvee X$ and $\bigwedge X$ exist for all subsets X of L.
(iii) A lattice $(L, \leq)$ is called conditionally complete if $\bigvee X$ and $\bigwedge X$ exist for all non-empty bounded subsets X of L. It is clear that every complete lattice is a conditionally complete lattice.
(iv) A lattice $(L, \leq)$ is called infinitely distributive if

$$a \wedge \left(\bigvee_{i \in I} x_i \right) = \bigvee_{i \in I} (a \wedge x_i), \tag{1}$$

and

$$a \vee \left(\bigwedge_{i \in I} x_i \right) = \bigwedge_{i \in I} (a \vee x_i). \tag{2}$$

where I is an index set.

Note that (1) and (2) may not hold in every complete distributive lattice. See the following example.

Example 2.3 ([1]) Let $(L, \subseteq)$ be the complete lattice of all closed subsets of the plane. Let c denote the circle $x^2 + y^2 = 1$ and d_k denote the disc $x^2 + y^2 \leq 1 - k^{-2}$, then $c \wedge (\bigvee_{k=1}^{\infty} d_k) = c$, but $\bigvee_{k=1}^{\infty} (c \wedge d_k)$ is the empty set. On the other hand, (2) holds in this lattice, because $\vee$ and $\wedge$ coincide with the set-theoretic operations $\cup$ and $\cap$, respectively.

Theorem 2.4 ([1], Theorem V.5.16) *In any complete Boolean lattice L, (1) and (2) hold for any index set I.*

Furthermore, in any complete lattice, (1) and (2) imply the following equalities (see [1]).

Lemma 2.5 *In any complete lattice,* (1) *implies that*

$$\left(\bigvee_{i \in I_1} x_i \right) \wedge \left(\bigvee_{j \in I_2} y_j \right) = \bigvee_{i \in I_1} \bigvee_{j \in I_2} \left(x_i \wedge y_j \right), \tag{3}$$

and (2) *implies*

$$\left(\bigwedge_{i \in I_1} x_i \right) \vee \left(\bigwedge_{j \in I_2} y_j \right) = \bigwedge_{i \in I_1} \bigwedge_{j \in I_2} \left(x_i \vee y_j \right). \tag{4}$$

Note that (3) and(4) are also true for finite number of index sets $I_1, \ldots, I_n$; by induction.

Corollary 2.6 *In any complete Boolean lattice,* (3) *and* (4) *hold.*

Definition 2.7 Let H be a non-empty set and $* : H \times H \longrightarrow H$ be a binary operation. Then

(i) $(H, *)$ is called a semigroup if $a * (b * c) = (a * b) * c, \forall a, b, c \in H$.
(ii) A semigroup $(H, *)$ is called a monoid if it contains an element $e \in H$ such that $e * a = a * e = a, \forall a \in H$.
(iii) A monoid $(H, *)$ is called a group if for every element $a \in H$, there exists an element $a^{-1} \in H$ such that $a * a^{-1} = a^{-1} * a = e$.

Definition 2.8 Let $(H, *)$ be a commutative group (semigroup, monoid, respectively) with a partial order $\leq$. Then $(H, *, \leq)$ is called a lattice-ordered commutative group (semigroup, monoid, respectively) if

$$a \leq b \Longrightarrow a * c \leq b * c, \quad \forall a, b, c \in H,$$

for simplification, we call it l-group (l-semigroup, l-monoid, respectively).

Example 2.9 (i) Every lattice $(L, \leq)$ is an l-semigroup by letting $* = \wedge$ or $* = \vee$. Clearly a bounded lattice is an l-monoid in the same way.

(ii) The additive group $(Z, +, \leq)$ of integers is an l-group; the same is true for the additive group $(Q, +, \leq)$ of rational numbers.

(iii) Let $G = C[0, 1]$ be the set of all continuous real-valued functions on closed interval $[0, 1]$. Then G is an l-group with respect to usual + and $\leq$ of real-valued functions.

Theorem 2.10 ([1], Theorem XIII.14.25) *In any conditionally complete l-group,* (1) *and* (2) *hold.*

Definition 2.11 Let $Mat_{n\times m}(L)$ be the set of all $n \times m$ matrices over a lattice $(L, \leq)$. Define a partial order relation on $Mat_{n\times m}(L)$ as follows:

$$X \leq Y \Leftrightarrow x_{ij} \leq y_{ij} \quad \forall i = 1, 2, \ldots, n, \forall j = 1, 2, \ldots, m,$$

where $X, Y \in Mat_{n\times m}(L)$.

One can see that $(Mat_{n\times m}(L), \leq)$ is a lattice where its supremum and infimum are defined componentwise induced by the supremum and infimum of lattice L, respectively.

Definition 2.12 A partially ordered set P satisfies the descending chain condition (DCC) when every nonvoid subset of P has a minimal element.

Clearly if a partially ordered set P satisfies the DCC, then so do all its subsets (under the same partial ordering).

Definition 2.13 (i) Any finite partially ordered set (and therefore any finite lattice) P satisfies the DCC.

(ii) Let N be the set of all natural numbers. Then it satisfies the DCC with respect to both usual order on N and divisibility relation.

(iii) Let $L \simeq N \times N \times \cdots \times N$ (n-times) and $\leq$ be the order induced by $\leq$ on N. Then L satisfies the DCC.

Definition 2.14 Let $(H, *, \leq)$ be an l-semigroup and A and X be $m \times n$ and $n \times 1$ matrices over H, respectively. Define the matrix $A * X$ by

$$A * X = \left(\bigvee_{j=1}^{m} a_{ij} * x_j \right)_{m\times 1}.$$

Definition 2.15 (i) Let $(L, \leq)$ be a lattice. Then, every non-zero minimal element of L is called an atom.

(ii) An atomic lattice is a lattice L in which every element is a join of atoms, and hence the join of the atoms which it contains.

(iii) A non-zero element $a \in L$ is called join-irreducible if $a = b \vee c$ implies $a = b$ or $a = c$. We denote the set of all join-irreducible elements of L by J.

(iv) For any $a \in L$, let $J(a) = \{j \in J | j \leq a\}$.

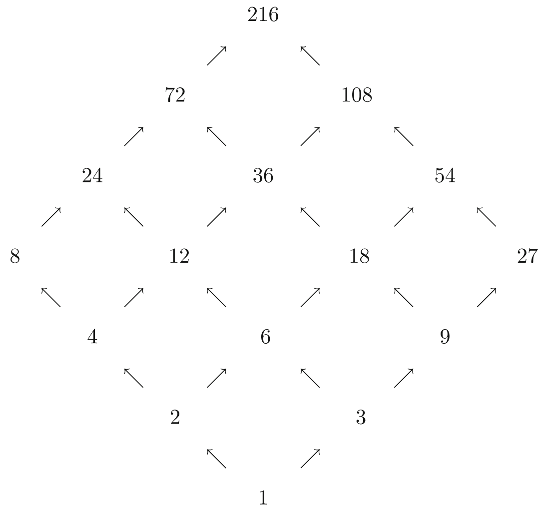

Fig. 1 Relationship between and among elements of L

Note that if L satisfies the DCC then every non-zero element $a \in L$ can be represented as a finite supremum of join-irreducible elements; i.e., $a = \bigvee J(a)$ (see [1,3]).

Example 2.16 (i) Let X be an arbitrary set. Then $P(X)$, the set of all subsets of X, is a Boolean atomic lattice.

(ii) Let $L = [0, 1]$ be the bounded chain of real numbers between 0 and 1. Then L is an atom-less lattice while every non-zero element $a \in L$ is join-irreducible.

(iii) Let $L = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 27, 36, 54, 72, 108, 216\}$ be the set of all divisors of 216 and $x \leq y$ means x divides y. The set of join-irreducible elements of L, which denotes by J, is $J = \{2, 3, 4, 8, 9, 27\}$ while L has only two atoms 2 and 3 (see Fig. 1).

(iv) (See [2]) Let $O_{reg}(R)$ be the set of all regular open subsets of real numbers, that is, those sets equal to interior of their closures. The sup is not the union of the regular open sets but the interior of the closure of their union. The inf is the interior of their intersection. Remarkably, $O_{reg}(R)$ is a complete Boolean lattice where the lattice complement of $U \in O_{reg}(R)$ is the interior of $R \backslash U$. Note that $O_{reg}(R)$ has neither atom nor join-irreducible element.

Throughout this paper supremum and infimum over the empty set ϕ are taken to be 0 and 1, respectively.

3 Solutions of $A * X \geq b$

The fuzzy linear system $A * X \geq b$ with $* = \wedge$, over a bounded chain has been studied by Peeva [8–10]. She found a necessary and sufficient computational condition for consistency of this system. She also gave an algorithm for finding its minimal solutions. In this section, we replace a bounded chain by a distributive lattice and we give a necessary and sufficient condition for consistency of this system. The approach given here, is an algebraic approach and totally different from previous one. But existence of minimal solution(s) not necessarily holds for any distributive lattice (see Example 3.1). We may also have a linear system such that it has a minimal solution and also has a chain of solutions which does not tend to a minimal solution. Therefore, it is very important to find a necessary and sufficient condition for existence of a minimal solution. We must be sure that for any solution X, there exists a minimal solution $\tilde{X}$ such that $\tilde{X} \leq X$. In Example 3.1 one can see both of the above situations, which can be classified as follows:

(i) A consistent linear inequality which has no minimal solution,
(ii) A consistent linear inequality which has a minimal solution and also has an infinite chain of solutions which does not tend to a minimal solution.
In this section we give some sufficient condition for existence of a minimal solution(s).

Let $(L, \leq)$ be a distributive lattice. By a linear system of inequalities $A * X \geq b$ over L we mean the following system of inequalities:

$$\begin{cases} (a_{11} \wedge x_1) \vee (a_{12} \wedge x_2) \vee \ldots \vee (a_{1n} \wedge x_n) \geq b_1 \\ (a_{21} \wedge x_1) \vee (a_{22} \wedge x_2) \vee \ldots \vee (a_{2n} \wedge x_n) \geq b_2 \\ \cdot \\ \cdot \\ \cdot \\ (a_{m1} \wedge x_1) \vee (a_{m2} \wedge x_2) \vee \ldots \vee (a_{mn} \wedge x_n) \geq b_m \end{cases} \tag{5}$$

where $a_{ij}, x_j, b_i \in L$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Example 3.1 Consider two chains C_1 and C_2, both isomorphic to half-open real interval (0, 1]. Let $L = C_1 \cup C_2 \cup \{0\}$. Define $\vee$ and $\wedge$ on L as follows:

$$a \vee b = \begin{cases} \max\{a, b\} & \text{if } a, b \in C_1 \text{ or } a, b \in C_2 \\ 1 & \text{otherwise} \end{cases}$$

and

$$a \wedge b = \begin{cases} \min\{a, b\} & \text{if } a, b \in C_1 \text{ or } a, b \in C_2 \\ 0 & \text{otherwise} \end{cases}$$

One can see that $(L, \vee, \wedge)$ is a bounded distributive lattice. Consider the linear inequality $(a \wedge x) \vee (b \wedge y) \geq 1$, where $a \in C_1$, $b \in C_2$ and $a, b < 1$. This inequality has infinitely many solutions (let $x \in C_1$ and $y \in C_2$), but does not have any minimal solution. Now consider the linear inequality $x \vee y \geq 1$. It has two minimal solutions; $x = 0, y = 1$ and $x = 1, y = 0$. Moreover, $x = a_i, y = b_i$, where $a_i \in C_1, b_i \in C_2$ and $a_i, b_i < 1$ for all i, is also a chain of solutions of $x \vee y \geq 1$. One can see it does not tend to a minimal solution. Note that L is not infinitely distributive lattice.

Theorem 3.2 ([6], Theorem 3.1) *Let $(L, *, \leq)$ be an l-semigroup, $A = (a_{ij})_{m\times n}$ and $X_i = (x_{ji})_{n\times 1}$; $i = 1, 2$, be matrices over L. Consider the partial order on $Mat_{n\times 1}(L)$ as in Definition* 2.11. *If $X_1 \leq X_2$ then $A * X_1 \leq A * X_2$.*

Corollary 3.3 *Let $(L, *, \leq)$ be an l-semigroup, $A = (a_{ij})_{m\times n}$ and $X_i = (x_{ji})_{n\times 1}$; $i = 1, 2$, be matrices over L. Then*

(i) $A * (X_1 \wedge X_2) \leq A * X_1 \wedge A * X_2$
(ii) $A * (X_1 \vee X_2) \geq A * X_1 \vee A * X_2$

Remark 3.4 Note that in Corollary 3.3(i), if $X_1 \leq X_2$, then $A*(X_1 \wedge X_2) = A*X_1 \wedge A*X_2$. Clearly if $\{X_j\}_{j=1}^{n}$ is a finite chain, then $A * (\bigwedge_{j=1}^{n} X_j) = \bigwedge_{j=1}^{n}(A * X_j)$.

A necessary and sufficient condition for the consistency of (5) is given in the following theorem.

Theorem 3.5 ([6], Theorem 4.1) *Let L be a distributive lattice. Let A, X and b be $m\times n$, $n\times 1$ and $m \times 1$ matrices over L, respectively. Let A_j be the jth column of A for $j = 1, 2, \ldots n$. The linear system* (5) *is consistent if and only if* $\bigvee_{j=1}^{n} A_j \geq b$.

Theorem 3.6 ([6]) *Let L, A, X and b be as in Theorem* 3.5. *If S is the set of feasible solutions of* (5)*, then S is a convex join-semi-sublattice of* L^n.

In any bounded chain, by the following theorem we surly have a minimal solution for (5) provided it is consistent.

Theorem 3.7 ([6], Theorem 4.3) *Let L be a bounded chain, A and b as in Theorem* 3.5. *If* (5) *is consistent, then it has a minimal solution.*

Now we investigate a condition for an arbitrary lattice such that under this condition we have surly a minimal solution for (5), if it is consistent.

Definition 3.8 Let $(L, \leq)$ be a distributive lattice and consider two arbitrary descending chains $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$ of elements of L. We say L satisfies infinite chains meet distributivity (ICMD) if

$$\bigwedge_{i\in I}(a_i \vee b_i) = \left(\bigwedge_{i\in I} a_i\right) \vee \left(\bigwedge_{i\in I} b_i\right). \tag{6}$$

Note that in any lattice

$$\bigwedge_{i\in I}(a_i \vee b_i) \geq \left(\bigwedge_{i\in I} a_i\right) \vee \left(\bigwedge_{i\in I} b_i\right) \tag{7}$$

holds.

Equality (6) also can be defined over an arbitrary lattice L and distributivity is not a necessary condition for (6). For example (6) holds in any finite lattice which may not be distributive (see Example 3.11). One can find a distributive lattice such that (6) does not hold (see Example 3.1). But to find minimal solutions we need distributivity of L (see Remark 4.4). Hence we defined (6) over a distributive lattice.

Theorem 3.9 *Let $(L, \leq)$ be a distributive lattice that satisfies* (4)*. Then L satisfies the ICMD.*

Proof Let $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$ be two arbitrary descending chains of elements of L. We have $(\bigwedge_{i\in I} a_i) \vee (\bigwedge_{i\in I} b_i) = \bigwedge_{i\in I}\bigwedge_{j\in I}(a_i \vee b_j)$, by (4). But $a_i \vee b_j \geq a_i \vee b_i \geq \bigwedge_{i\in I}(a_i \vee b_i)$, $\forall j$ and $\forall i \geq j$. Hence $\bigwedge_{i\in I}\bigwedge_{j\in I}(a_i \vee b_j) \geq \bigwedge_{i\in I}(a_i \vee b_i)$. Now equality holds by (7). □

Note that the ICMD is a weaker condition than (2).

Example 3.10 (i) A distributive lattice which satisfies the DCC, satisfies the ICMD, too.
(ii) Any infinitely distributive lattice, satisfies the ICMD.
(iii) Any conditionally complete l-group satisfies (1) and (2) by Theorem 2.10, hence it satisfies the ICMD.

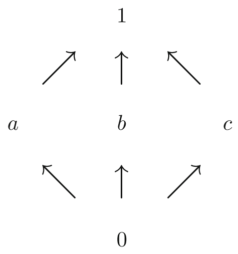

Fig. 2

Example 3.11 Let $L = \{0, a, b, c, 1\}$ be a diamond. The relationship of these elements is shown in Fig. 2.

Clearly, L is not a distributive lattice, however (6) holds.

Theorem 3.12 *Let $(L, \leq)$ be an arbitrary lattice. Consider the linear system of inequalities* (5) *over L. Then, any chain of solutions of* (5) *tends to a minimal solution if and only if L satisfies the ICMD.*

Proof Suppose L satisfies the ICMD. Let S be the set of all feasible solutions of (5) and $\{X_j\}_{j\in J}$ be a descending chain in S. We have $\bigvee_{k=1}^{n}(a_{ik} \wedge x_{kj}) \geq b_i$ for $i = 1, 2, \ldots, m$ and for all $j \in J$. Hence $\bigwedge_{j\in J} \bigvee_{k=1}^{n}(a_{ik} \wedge x_{kj}) \geq b_i$, for $i = 1, 2, \ldots, m$. Therefore, by (6), $\bigvee_{k=1}^{n}(a_{ik} \wedge (\bigwedge_{j\in J} x_{kj})) \geq b_i$, for $i = 1, 2, \ldots, m$, which means that $X = \bigwedge_{j\in J} X_j \in S$. Existence of a minimal solution follows from Zorn's Lemma.

Now, suppose L does not satisfy the ICMD. Hence there exist two descending chains $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$ such that $\bigwedge_{i\in I}(a_i \vee b_i) > (\bigwedge_{i\in I} a_i) \vee (\bigwedge_{i\in I} b_i)$. Consider the inequality $x \vee y \geq \bigwedge_{i\in I}(a_i \vee b_i)$. Obviously, $(a_i, b_i)_{i\in I}$ is a descending chain of solutions of this inequality which does not tend to a minimal solution. □

Corollary 3.13 *Let $(L, \leq)$ be a lattice which satisfies the DCC. Then,* (5) *has a minimal solution provided it is consistent.*

Proof It follows from Example 3.10(i) and Theorem 3.12. □

Corollary 3.14 *Let $(L, \wedge, \leq)$ be a conditionally complete l-group. If* (5) *is consistent, then it has a minimal solution.*

Proof It follows from Theorem 2.10, Theorem 3.9 and Theorem 3.12. □

Remark 3.15 For finding a minimal solution of the linear system (5), it is very important to find a minimal solution for each of inequalities (see Algorithm 4.7). Consider the following consistent inequality over a distributive lattice L:

$$\bigvee_{i=1}^{n}(a_i \wedge x_i) \geq b \tag{8}$$

Suppose (8) has a minimal solution and $X = (x_i)_{n\times 1}$ to be such a solution. Since X satisfies (8) then $(\bigvee_{i=1}^{n}(a_i \wedge x_i)) \wedge b = b$, which implies $(\bigvee_{i=1}^{n}(a_i \wedge x_i \wedge b)) = b$. Let $y_i = x_i \wedge b$ for $i = 1, 2, \ldots, n$, then $(\bigvee_{i=1}^{n}(a_i \wedge y_i)) = b$. Hence, $Y = (y_i)_{n\times 1}$ is a solution of (8). But $y_i = x_i \wedge b \leq x_i$ for $i = 1, \ldots, n$, hence $Y = X$ by minimality of X and $(\bigvee_{i=1}^{n}(a_i \wedge x_i)) = (\bigvee_{i=1}^{n}(a_i \wedge y_i)) = b$. Therefore, equality holds for all minimal solutions of (8). Moreover, if $X = (x_i)_{n\times 1}$ is a minimal solution of (8), then $x_i \leq b$ for $i = 1, \ldots, n$. Let $\tilde{S}$ be a subset of S (the set of solutions of (8)) such that the equality holds for all elements of $\tilde{S}$. Now, it is clear that minimal solutions of (8) are minimal elements of $\tilde{S}$.

4 Finding a minimal solution

In this section we are looking for all minimal solutions of the linear system (5). To do this, as it was mentioned in Remark 3.15, it is necessary to find all minimal solutions of any inequality in the system one by one. Finding minimal solutions of inequality (8) completely depends on the structure of the lattice. For example, if L is a bounded chain, it is very easy to find them; just consider a coefficient a_i to be greater than or equal b, let $x_i = b$ and the others equal to zero (see [8–10]). But in other lattices it is not that easy. The following remark gives an upper bound for all minimal solutions and Theorem 4.2 helps us to find them up.

Remark 4.1 Suppose $X = (x_i)_{n\times 1}$ is a minimal solution of (8) over a distributive lattice L. By Remark 3.15, $\bigvee_{i=1}^{n}(a_i \wedge x_i) = b$. Clearly, $Y = (y_i)_{n\times 1}$ is a solution of (8) where $y_i = a_i \wedge x_i$ for $i = 1, \ldots, n$ and $Y \leq X$. Hence, $Y = X$ by minimality of X, which means $x_i = a_i \wedge x_i$ for $i = 1, \ldots, n$. Therefore, $x_i \leq a_i$ for $i = 1, ..., n$. Furthermore, $x_i \leq b$ for $i = 1, \ldots, n$, by Remark 3.15. Hence, $x_i \leq a_i \wedge b$ for $i = 1, \ldots, n$.

Theorem 4.2 *Consider the following two systems:*

$$\begin{cases} \bigvee_{i=1}^{n}(a_i \wedge x_i) = b \\ x_i \leq a_i \wedge b \quad i = 1, \ldots, n \end{cases} \tag{9}$$

and

$$\begin{cases} \bigvee_{i=1}^{n} x_i = b \\ x_i \leq a_i \wedge b \quad i = 1, \ldots, n \end{cases} \tag{10}$$

Then $X = (x_i)_{n\times 1}$ is a solution of (9) if and only if it is a solution of (10).

The proof is straight forward.

Corollary 4.3 *Minimal solutions of linear inequality* (8) *coincide with minimal solutions of the linear system* (10).

Proof By Remark 4.1, $X = (x_i)_{n\times 1}$ is an upper bound of all minimal solutions of (8) where $x_i = a_i \wedge b, i = 1, \ldots, n$ and equality holds for all minimal solutions. Hence, they are the solutions of (9) which are equivalent to (10). Therefore, minimal solutions of (8) are the same as the minimal solutions of (10). □

Remark 4.4 (i) Finding a minimal solution of (8) is completely depends on the structure of the lattice. In distributive lattices by using Remark 3.15 we have a powerful criterion for finding them (see Theorem 4.2 and Corollary 4.3). Note that Theorem 4.2 and Corollary 4.3 may not hold on non-distributive lattice. For example, let L be the lattice in Example 3.11 and consider the inequality $(a \wedge x) \vee (c \wedge y) \geq b$. However, this inequality has a minimal solution, $x = a$, $y = c$; but equality does not hold for this minimal solution.
(ii) The inequality (8) over a distributive lattice L, may have infinitely many minimal solutions. For example let $L = P(R)$ be the Boolean lattice of all subsets of real numbers. Consider the inequality

$$((1, 2) \wedge x_1) \vee ((1.9, 4) \wedge x_2) \geq (1.5, 3) \tag{11}$$

over L. Note that $X_\epsilon = ((1.5, 1.9+\epsilon), [1.9+\epsilon, 3))^T$ as well as $X_\delta = ((1.5, 1.9+\delta], (1.9+\delta, 3))^T$ for $\epsilon \in (0, 0.1]$ and $\delta \in [0, 0.1)$ are minimal solutions of (11), where "T" is the transpose operation. Therefore, (11) has infinitely many minimal solutions.

Example 4.5 Let L be the lattice in Example 2.16 (iii) and consider the inequality

$$(4 \wedge x_1) \vee (6 \wedge x_2) \vee (3 \wedge x_3) \geq 6 \tag{12}$$

By Remark 4.1, $X = (2, 6, 3)^T$ is an upper bound of minimal solutions of (12), and minimal solutions of (12) are the minimal solutions of (10) with $x_1 \leq 2, x_2 \leq 6, x_3 \leq 3$ and $b = 6$. One can find four minimal solutions for (12); $X_{m_1} = (1, 6, 1)^T$, $X_{m_2} = (2, 1, 3)^T$, $X_{m_3} = (2, 3, 1)^T$ and $X_{m_4} = (1, 2, 3)^T$.
Now consider another inequality

$$(8 \wedge x_1) \vee (27 \wedge x_2) \vee (9 \wedge x_3) \geq 36 \tag{13}$$

The inequality (13) has just two minimal solutions $X_{m_1} = (4, 9, 1)^T$ and $X_{m_2} = (4, 1, 9)^T$.

Example 4.6 Let $L = P(X)$ be the Boolean lattice of all subsets of $X = \{a, b, c, d\}$ and consider the following inequality:

$$(\{a, b, d\} \cap x_1) \cup (\{c\} \cap x_2) \cup (\{a, c, d\} \cap x_3) \geq \{b, c, d\} \tag{14}$$

By Remark 4.1, $X = (\{b, d\}, \{c\}, \{c, d\})^T$ is an upper bound of minimal solutions. Now solve (10) for this upper bound and obtain four minimal solutions $X_{m_1} = (\{b\}, \{c\}, \{d\})^T$, $X_{m_2} = (\{b\}, \phi, \{c, d\})^T$, $X_{m_3} = (\{b, d\}, \{c\}, \phi)^T$ and $X_{m_4} = (\{b, d\}, \phi, \{c\})^T$.

Finding minimal solutions of an inequality is different from finding them over a chain, but if the number of minimal solutions of (8) is finite, then the number of minimal solutions for a linear system of inequalities (5) is finite and finding them is almost the same as it is over a chain. If n_i denotes the number of minimal solutions of ith inequality for $i = 1, \ldots, n$, then $n = \Pi_{i=1}^n n_i$ is an upper bound for the number of minimal solutions of (5). In Algorithm 4.7 and its implementation we use Peeva's approach and notations (see [10]).

Algorithm 4.7

1. Find all minimal solutions of all inequalities individually by using Corollary 4.3.
2. Show any minimal solution by $\prod_{1\leq j\leq n}\langle\frac{x_j}{j}\rangle$, which means the jth component of solution is equal to x_j. Note that the product means "and". For example, the solution $X = (2, 1, 3)^T$ is shown by $\langle\frac{2}{1}\rangle.\langle\frac{1}{2}\rangle.\langle\frac{3}{3}\rangle$.
3. Let $W_i = \sum\prod_{1\leq j\leq n}\langle\frac{x_j}{j}\rangle$ where summation is taken over all minimal solutions of the ith inequality. Note that the sum means "or".
4. We have to consider all inequalities simultaneously. Therefore, minimal solutions of the system are computed by the concatenation

$$W = \prod_{1\leq i\leq m} W_i. \tag{15}$$

5. Expanding the parentheses in (15) by using the following properties of the concatenation. We obtain a set of ways, from which we extract minimal solutions:

$$W = \sum\prod_{1\leq j\leq n}\left\langle\frac{x_j}{j}\right\rangle \tag{16}$$

In order to compute minimal solutions of (5), it is important to determine different ways to satisfy simultaneously inequalities of the system. To achieve this aim we are interested in properties of the concatenation (15).

Properties of the concatenation

1. In the concatenation, multiplication is distributive with respect to addition, i.e.,

$$\left\langle \frac{a}{j_1} \right\rangle \left(\left\langle \frac{b}{j_2} \right\rangle + \left\langle \frac{c}{j_3} \right\rangle \right) = \left\langle \frac{a}{j_1} \right\rangle . \left\langle \frac{b}{j_2} \right\rangle + \left\langle \frac{a}{j_1} \right\rangle . \left\langle \frac{c}{j_3} \right\rangle . \tag{17}$$

2. This property is called absorption for multiplication:

$$\left\langle \frac{a}{j} \right\rangle . \left\langle \frac{b}{j} \right\rangle = \left\langle \frac{a \vee b}{j} \right\rangle . \tag{18}$$

   The above expression gives a minimal solution for two different inequalities.
3. Concatenation (15) is commutative with respect to addition and multiplication.
4. This property is called absorption for addition:

$$\prod_{1 \leq j \leq n} \left\langle \frac{x_j}{j} \right\rangle + \prod_{1 \leq j \leq n} \left\langle \frac{y_j}{j} \right\rangle = \prod_{1 \leq j \leq n} \left\langle \frac{x_j}{j} \right\rangle , \tag{19}$$

if $x_j \leq y_j$ for $j = 1, \ldots, n$.

Example 4.8 Let L be the lattice of Example 4.5 and consider the following linear system of inequalities over L:

$$\begin{cases} (4 \wedge x_1) \vee (6 \wedge x_2) \vee (3 \wedge x_3) \geq 6 \\ (8 \wedge x_1) \vee (27 \wedge x_2) \vee (9 \wedge x_3) \geq 36 \\ (3 \wedge x_1) \vee (9 \wedge x_2) \vee (4 \wedge x_3) \geq 18 \end{cases} \tag{20}$$

We found minimal solutions of the first and the second inequalities in Example 4.5. The only one minimal solution of the last inequality is $(1, 9, 2)^T$. We have:

$$\begin{aligned} W_1 &= \left\langle \frac{1}{1} \right\rangle . \left\langle \frac{6}{2} \right\rangle . \left\langle \frac{1}{3} \right\rangle + \left\langle \frac{2}{1} \right\rangle . \left\langle \frac{1}{2} \right\rangle . \left\langle \frac{3}{3} \right\rangle + \left\langle \frac{2}{1} \right\rangle . \left\langle \frac{3}{2} \right\rangle . \left\langle \frac{1}{3} \right\rangle + \left\langle \frac{1}{1} \right\rangle . \left\langle \frac{2}{2} \right\rangle . \left\langle \frac{3}{3} \right\rangle \\ W_2 &= \left\langle \frac{4}{1} \right\rangle . \left\langle \frac{1}{2} \right\rangle . \left\langle \frac{9}{3} \right\rangle + \left\langle \frac{4}{1} \right\rangle . \left\langle \frac{9}{2} \right\rangle . \left\langle \frac{1}{3} \right\rangle \\ W_3 &= \left\langle \frac{1}{1} \right\rangle . \left\langle \frac{9}{2} \right\rangle . \left\langle \frac{2}{3} \right\rangle \end{aligned}$$

To compute $W = W_1.W_2.W_3$, first compute $W_2.W_3$ as follows:

$$W_2.W_3 = \left(\left\langle\frac{4}{1}\right\rangle.\left\langle\frac{1}{2}\right\rangle.\left\langle\frac{9}{3}\right\rangle + \left\langle\frac{4}{1}\right\rangle.\left\langle\frac{9}{2}\right\rangle.\left\langle\frac{1}{3}\right\rangle\right).\left(\left\langle\frac{1}{1}\right\rangle.\left\langle\frac{9}{2}\right\rangle.\left\langle\frac{2}{3}\right\rangle\right)$$
$$= \left\langle\frac{4}{1}\right\rangle.\left\langle\frac{9}{2}\right\rangle.\left\langle\frac{18}{3}\right\rangle + \left\langle\frac{4}{1}\right\rangle.\left\langle\frac{9}{2}\right\rangle.\left\langle\frac{2}{3}\right\rangle = \left\langle\frac{4}{1}\right\rangle.\left\langle\frac{9}{2}\right\rangle.\left\langle\frac{2}{3}\right\rangle$$

Now

$$W = \left(\left\langle\frac{1}{1}\right\rangle.\left\langle\frac{6}{2}\right\rangle.\left\langle\frac{1}{3}\right\rangle + \left\langle\frac{2}{1}\right\rangle.\left\langle\frac{1}{2}\right\rangle.\left\langle\frac{3}{3}\right\rangle + \left\langle\frac{2}{1}\right\rangle.\left\langle\frac{3}{2}\right\rangle.\left\langle\frac{1}{3}\right\rangle\right.$$
$$\left. + \left\langle\frac{1}{1}\right\rangle.\left\langle\frac{2}{2}\right\rangle.\left\langle\frac{3}{3}\right\rangle\right).\left(\left\langle\frac{4}{1}\right\rangle.\left\langle\frac{9}{2}\right\rangle.\left\langle\frac{2}{3}\right\rangle\right) = \left\langle\frac{4}{1}\right\rangle.\left\langle\frac{18}{2}\right\rangle.\left\langle\frac{2}{3}\right\rangle$$
$$+ \left\langle\frac{4}{1}\right\rangle.\left\langle\frac{9}{2}\right\rangle.\left\langle\frac{6}{3}\right\rangle + \left\langle\frac{4}{1}\right\rangle.\left\langle\frac{9}{2}\right\rangle.\left\langle\frac{2}{3}\right\rangle + \left\langle\frac{4}{1}\right\rangle.\left\langle\frac{18}{2}\right\rangle.\left\langle\frac{6}{3}\right\rangle = \left\langle\frac{4}{1}\right\rangle.\left\langle\frac{9}{2}\right\rangle.\left\langle\frac{2}{3}\right\rangle$$

Therefore, the linear system (20) has exactly one minimal solution:

$$x_1 = 4, \quad x_2 = 9, \quad x_3 = 2$$

Example 4.9 Let L be the lattice of Example 4.6 and consider the following linear system of inequalities over L:

$$\begin{cases} (\{a, b, d\} \cap x_1) \cup (\{c\} \cap x_2) \cup (\{a, c, d\} \cap x_3) \geq \{b, c, d\} \\ (\{c, d\} \cap x_1) \cup (\{b, d\} \cap x_2) \cup (\{a, b, c\} \cap x_3) \geq \{a, c\} \end{cases} \tag{21}$$

We found minimal solutions of the first inequality in Example 4.5. The second inequality has two minimal solutions; $(\{c\}, \phi, \{a\})^T$ and $(\phi, \phi, \{a, c\})^T$. By applying Algorithm 4.7, we get four minimal solutions for (21) as follows:

$$X_{m_1} = (\{b, c\}, \{c\}, \{a, d\})^T, \quad X_{m_2} = (\{b\}, \phi, \{a, c, d\})^T$$
$$X_{m_3} = (\{b, c, d\}, \{c\}, \{a\})^T, \quad X_{m_4} = (\{b, d\}, \phi, \{a, c\})^T$$

Remark 4.10 Consider the linear system of equalities

$$A * X = b \tag{22}$$

over a distributive lattice L. If (22) is consistent then minimal solutions of it can be computed the same as (5). A theoretical necessary and sufficient condition for consistency of (22) is given in [6] and it is interesting to find a computational one.

5 The optimization problem

In this section we seek an optimal solution of the linear system of inequalities (5) with respect to a linear objective function. By a linear objective function $Z = C * X$, we mean $C * X = \bigvee_{i=1}^{n}(c_i \wedge x_i)$, where $C = (c_i)_{1\times n}$ and $X = (x_i)_{n\times 1}$ are matrices over L. We show this problem by

$$\min\{Z = C * X | A * X \geq b\}. \tag{23}$$

Clearly $C * X \leq C * Y$, whenever $X \leq Y$ by Theorem 3.2. Therefore, an optimal solution for minimizing the objective function Z is among minimal solutions of (5). In this section we give an upper bound for all minimal solutions of (5). Then, by means of this upper bound and objective function, we will find the minimum of Z without finding all minimal solutions of (5). Also, we offer a solution X for (5) such that the minimal solution which is less than or equal X, is the optimal solution. The Algorithm 5.2 can be used even if (5) to have infinitely many minimal solutions. This algorithm totally depends on join-irreducible elements (see Definition 2.15(iii)). Hence, it is suitable for distributive lattices which every element of it to be the supremum of join-irreducible elements. For example, if L satisfies the DCC, then every element of L is a finite supremum of join-irreducible elements (Also see Remark 5.6(i)). This algorithm does not work on chains but by a few changes it works on a Boolean lattice even it has not join-irreducible elements (see Example 2.16 (iv)). Note that the complement of $J(a)$ (see Definition 2.15(iv)) is denoted by $J(a)^c$ and it is equal to $J \backslash J(a)$. The following example may help to understand how the algorithm works.

Example 5.1 Consider the optimization problem (23) where the linear system of inequalities (5) is as in Example 4.9, and $C_1 = (\{a\}, \{a, c\}, \{a, d\})$. Then

$$C_1 * X_{m_1} = (\{a\} \cap \{b, c\}) \cup (\{a, c\} \cap \{c\}) \cup (\{a, d\} \cap \{a, d\}) = \{a, c, d\}$$
$$C_1 * X_{m_2} = (\{a\} \cap \{b\}) \cup (\{a, c\} \cap \phi) \cup (\{a, d\} \cap \{a, c, d\}) = \{a, d\}$$
$$C_1 * X_{m_3} = (\{a\} \cap \{b, c, d\}) \cup (\{a, c\} \cap \{c\}) \cup (\{a, d\} \cap \{a\}) = \{a, c\}$$
$$C_1 * X_{m_4} = (\{a\} \cap \{b, d\}) \cup (\{a, c\} \cap \phi) \cup (\{a, d\} \cap \{a, c\}) = \{a\}$$

The minimum of $Z = C_1 * X$ is $\{a\}$ and the optimal solution is X_{m_4}. Note that the supremum of all minimal solutions is equal to $\{a, b, c, d\}$ and it is possible to find an arrangement of atoms (or join-irreducible elements) to minimize Z. Now consider $C_2 = (\{b, d\}, \{c\}, \{d\})$. Then

$$C_2 * X_{m_1} = (\{b, d\} \cap \{b, c\}) \cup (\{c\} \cap \{c\}) \cup (\{d\} \cap \{a, d\}) = \{b, c, d\}$$
$$C_2 * X_{m_2} = (\{b, d\} \cap \{b\}) \cup (\{c\} \cap \phi) \cup (\{d\} \cap \{a, c, d\}) = \{b, d\}$$
$$C_2 * X_{m_3} = (\{b, d\} \cap \{b, c, d\}) \cup (\{c\} \cap \{c\}) \cup (\{d\} \cap \{a\}) = \{b, c, d\}$$
$$C_2 * X_{m_4} = (\{b, d\} \cap \{b, d\}) \cup (\{c\} \cap \phi) \cup (\{d\} \cap \{a, c\}) = \{b, d\}$$

In this case we have two optimal solutions; X_{m_2} and X_{m_4}, with same values for minimizing Z.

Algorithm 5.2 Consider the optimization problem (23) and the linear system of inequalities (5) over a bounded below distributive lattice L with 0 as the least element .

(i) Let $U_j = \bigvee_{i=1}^{m}(a_{ij} \wedge b_i)$ for $j = 1, \ldots, n$. Then U_j is an upper bound for jth component of all minimal solutions of (5).

(ii) Put $c_j = c_j \wedge U_j$ for $j = 1, \ldots, n$. This is a helpful reduction on objective function's coefficients because $c_j \wedge x_j \leq c_j \wedge U_j$ for $j = 1, \ldots, n$ and for all minimal solutions $X = (x_j)_{n \times 1}$.

(iii) 1. For $i = 1, \ldots, n$
2. For $j = 1, \ldots, n$
3. Put $J(U_i) = J(U_i) \cap J(U_i \wedge U_j \wedge c_i)^c$
4. Put $U_i = \bigvee J(U_i)$
5. Put $J(U_j) = J(U_j) \cap J(U_i \wedge U_j \wedge c_j)^c$
6. Put $U_j = \bigvee J(U_j)$
7. Next j
8. Next i
9. Put $Z = \bigvee_{j=1}^{n}(c_j \wedge U_j)$

10. The minimal solution which is less than or equal $U = (U_j)_{n\times 1}$ is an optimal solution for (23).

Remark 5.3 Note that in step (iii) of Algorithm 5.2, we reduced $c_j \wedge U_j$ as less as possible. Therefore, the value of Z which is computed in number 8 of step (iii) is the minimum value.

Example 5.4 Consider the optimization problem in Example 5.1. We want to find the optimal value of Z and optimal solution X, by following Algorithm 5.2.
Step (i): In this step we compute an upper bound for each component of a minimal solution. We have:

$$U_1 = \{b, c, d\}, \quad U_2 = \{c\}, \quad U_3 = \{a, c, d\}$$

Step (ii): Coefficients of the first objective function reduced as follows:

$$c_{11} = \phi, \quad c_{12} = \{c\}, \quad c_{13} = \{a, d\}$$

Step (iii): After computing new upper bounds we have:

$$U_1 = \{b, c, d\}, \quad U_2 = \phi, \quad U_3 = \{a, c\}$$

and $Z = \{a\}$. Note that X_{m_4} is the only solution which is less than the above upper bound.

Example 5.5 Let L be the lattice in Example 2.16 (iii) and consider the following inequality over L

$$(36 \wedge x_1) \vee (24 \wedge x_2) \vee (6 \wedge x_3) \geq 12 \tag{24}$$

There are six minimal solutions for the above inequality with upper bounds

$$U_1 = 12, \quad U_2 = 12, \quad U_3 = 6$$

which are listed in Table 1. Now consider an objective function with

$$c_1 = 18, \quad c_2 = 24, \quad c_3 = 27$$

One can reduce c_j, $j = 1, 2, 3$ to

$$c_1 = 6, \quad c_2 = 12, \quad c_3 = 3$$

By following Algorithm 5.2 one gets new upper bounds as follows

$$U_1 = 4, \quad U_2 = 1, \quad U_3 = 6$$

The minimum value of Z is 6 and the minimal solution which is less than or equal U is $X_{m_3} = (4, 1, 3)^T$. Note that there exist two other minimal solutions with the same value for Z.

Remark 5.6 (i) It is interesting to find a characterization for lattices which the Algorithm 5.2 is suitable for them.
(ii) The Algorithm 5.2 can be used for a Boolean lattice with a few changes, even it has no join-irreducible elements. The following algorithm mentions these changes.

Algorithm 5.7 Consider the optimization problem (23) and the linear system of inequalities (5) on a Boolean lattice L with 0 as the least element.

Table 1 Minimal solution of (24)

No	x_1	x_2	x_3	Z
1	1	12	1	12
2	12	1	1	6
3	4	1	3	6
4	1	4	3	12
5	3	4	1	12
6	4	3	1	6

(i) Let $U_j = \bigvee_{i=1}^{m}(a_{ij} \wedge b_i)$ for $j = 1, \ldots, n$.
(ii) Put $c_j = c_j \wedge U_j$ for $j = 1, \ldots, n$.
(iii) 1. For $i = 1, \ldots, n$
2. For $j = 1, \ldots, n$
3. Put $U_i = U_i \wedge (U_i \wedge U_j \wedge c_i)^c$
4. Put $U_j = U_j \wedge (U_i \wedge U_j \wedge c_j)^c$
5. Next j
6. Next i
7. Put $Z = \bigvee_{j=1}^{n}(c_j \wedge U_j)$
8. The minimal solution which is less than or equal $U = (U_j)_{n\times 1}$ is an optimal solution for (23).

Example 5.8 Let $L = O_{reg}(R)$ be the lattice in Example 2.16 (iv) and consider inequality (11) over L. This inequality over L has infinitely many minimal solutions, $X_\epsilon = ((1.5, 1.9+\epsilon), [1.9+\epsilon, 3))^T$ for $\epsilon \in (0, 0.1]$ and $X_\delta = ((1.5, 1.9+\delta], (1.9+\delta, 3))^T$ for $\delta \in [0, 0.1)$. Now consider the objective function Z with $c_1 = (1, 2)$ and $c_2 = (2, 3)$. Follow the Algorithm 5.7. In step (i) we have:

$$U_1 = (1.5, 2), \quad U_2 = (1.9, 3)$$

In step (ii) we reduce c_js to $c_1 = (1.5, 2)$ and $c_2 = (2, 3)$. In step (iii) we reduce upper bounds to

$$U_1 = (1.5, 1.9], \quad U_2 = (1.9, 3)$$

Therefore, $Z = (1.5, 1.9] \cup (2, 3)$ and the optimal solution is $X_0 = ((1.5, 1.9], (1.9, 3))^T$. Although the linear system (11) has infinitely many solutions, it has just one optimal solution for the optimization problem (23).

6 Conclusion

Solving fuzzy linear systems $A * X \leq b$, $A * X \geq b$ and $A * X = b$ over a bounded chain has many applications. For example, in fuzzy automata theory, in algebra for solving fuzzy matrix equations and inequalities as well as for relation equations, and inclusions and in fuzzy programming. This paper extends this concept to L-fuzzy linear systems. The paper is a discussion of methods for solving a minimization problem $min\{X|A * X \geq b\}$ over a bounded distributive lattice, where A, b are given matrices and all entries of X are unknown. Necessary and sufficient conditions are established for the existence of minimal solutions and the corresponding algorithm developed in this study.

This discussion is generalized for the linear programming problem $\min\{C * X | A * X \geq b\}$, where C is given. The corresponding algorithms were given in special cases (see Remark 5.6 (i)).

Further Research

(i) It is interesting to find a necessary and sufficient condition for lattice L, such that any optimization problem (23) over L has an optimal solution.
(ii) It is also interesting search for an algorithm which can find all optimal solutions of (23).
(iii) Is there any relationship among different values of Z?
(iv) Do Algorithms work correctly if we replace $\vee$ and $\wedge$ by any t-norm and t-conorm?
(v) It is also interesting to find a necessary and sufficient condition for consistency of a non-linear system of equality or inequalities over a bounded distributive lattice.

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